

Remarks on quotient algebras

Ryszard Frankiewicz

Sławomir Szczepaniak

Abstract

The structure of quotient Boolean algebras in terms of cardinal invariants is investigated. Some results of Gitik and Shelah regarding atomless ideals are reproved and proofs are significantly simplified.

1. Introduction

While studying the structure of Boolean algebras one faces to the problem of classifying them up to (boolean) isomorphism. There are several ways of distinguishing between Boolean algebras, e.g. one can try to show that they produce (elementarily) nonequivalent Boolean extensions. This approach was used by Gitik and Shelah in [3]. Although elegant their method is rather indirect and hides a very reason why the given Boolean algebras are not isomorphic or equivalent (c.f. below). It is our opinion that more traditional way - by invariants - is better suited for the mentioned problem. The presented paper follows this approach. As a result we obtain not only much simpler proofs of Gitik-Shelah's results but also (as we believe) more illuminating. The main contribution are elementary (classical) proofs of **Corollary 3** and **Corollary 4**.

For undefined notions we refer to [4] and fix those most frequent in the paper or slightly different from the mentioned source. By an *ideal* on a given set X we shall understand the family $\mathcal{I} \subseteq P(X)$ such that

- if $A_0, A_1 \in \mathcal{I}$ and $A \subseteq A_0 \cup A_1$ then $A \in \mathcal{I}$;
- $X \notin \mathcal{I}$ and $\{x\} \in \mathcal{I}$ for each $x \in X$.

We denote by $\mathcal{I}^+ := \{A \subseteq X : A \notin \mathcal{I}\}$ a family of \mathcal{I} -positive sets and by $\mathcal{I}^c := \{A \subseteq X : X \setminus A \in \mathcal{I}\}$ a family of \mathcal{I} -full measure sets. An ideal $\mathcal{I} \restriction Y := \{A \subseteq X : A \cap Y \in \mathcal{I}\}$ is called *restriction* of \mathcal{I} to the set B . The *additivity number* of \mathcal{I} is defined as

$$\text{add}(\mathcal{I}) := \min \left\{ \kappa : \text{there is } \mathcal{F} \subseteq \mathcal{I} \text{ of size } \kappa \text{ such that } \bigcup \mathcal{F} \in \mathcal{I}^+ \right\}.$$

An important feature of the additivity number is that it is always regular cardinal.

We are mostly focused on the quotient Boolean algebra $P(X)/\mathcal{I}$ with the ordering

$$[A] \leq [B] \quad \text{iff} \quad A \leq_{\mathcal{I}} B \quad \text{iff} \quad A \setminus B \in \mathcal{I}.$$

There are several properties and characteristics of \mathcal{I} formulated in terms of boolean structure of $P(X)/\mathcal{I}$. We say that an ideal \mathcal{I} on X is *atomless* if $P(X)/\mathcal{I}$ is an atomless Boolean algebras. In other words for no $A \in \mathcal{I}^+$ a restriction $\mathcal{I} \upharpoonright A$ is a *prime* ideal, i.e. $P(X) = (\mathcal{I} \upharpoonright A) \cup (\mathcal{I} \upharpoonright A)^c$. The paper deals only with atomless ideals as they are the most important, they produce nontrivial boolean extensions. In the latter case, i.e. if two Boolean algebras have isomorphic dense subsets, we call them *equivalent* (as forcing notions). The other idea connecting an ideal and its quotient Boolean algebra. is the notion of saturation. We say that an ideal $\mathcal{I} \subseteq P(X)$ is κ -*saturated* if there is no \mathcal{I} -*antichain* of size κ in $P(X)/\mathcal{I}$, i.e. if $\mathcal{A} \subseteq P(X)/\mathcal{I}$ and we have $a_0 \cdot a_1 = \mathbf{0}$ for all $a_0, a_1 \in \mathcal{A}$ then $|\mathcal{A}| < \kappa$. More generally, we define the *saturation number* of a given Boolean algebra. \mathbb{B} as

$$\text{sat}(\mathbb{B}) := \min \{ \kappa : \mathbb{B} \text{ is } \kappa\text{-saturated} \}.$$

This is connected to a notion of regular subalgebra. A subalgebra \mathbb{C} of \mathbb{B} (with operation inherited from \mathbb{B}) is called regular if maximal antichain in \mathbb{C} is also maximal in \mathbb{B} . To state a formulation of the next cardinal invariant recall that a family $\mathcal{D} \subseteq \mathbb{B}^+ := \mathbb{B} \setminus \{\mathbf{0}\}$ is *dense* in \mathbb{B} if for all $b \in \mathbb{B}^+$ one can find $d \in \mathcal{D}$ such that $d \leq_{\mathbb{B}} b$. We define the *density number* of \mathbb{B} (see [6]) as follows

$$\pi(\mathbb{B}) := \min \{ |\mathcal{D}| : \mathcal{D} \text{ is dense in } \mathbb{B} \}.$$

The density of an ideal can be seen as the strong notion of saturation, for $\text{sat}(\mathbb{B}) \leq \pi(\mathbb{B})^+$. As our proofs are inspired by *Base Matrix Lemma* ([1], 1.12) we need some notations concerning trees and distributivity. By a tree \mathcal{T} we shall understand a set of sequences of ordinals with the following ordering (" \smallfrown " denotes concatenation)

$$s \leq_{\mathcal{T}} t \quad \text{iff} \quad s = t \smallfrown w \quad \text{for some sequence of ordinals } w$$

with the largest element being \emptyset (empty sequences). An element $t \in \mathcal{T}$ is called *splitting node* (we say that t splits) if in \mathcal{T} there are two $\leq_{\mathcal{T}}$ -incomparable extensions of t . We write $\mathcal{T} \upharpoonright \alpha$ for a subtree of \mathcal{T} such that

$$\mathcal{T} \upharpoonright \alpha = \bigcup_{\beta < \alpha} \text{Lev}_{\beta}(\mathcal{T})$$

where $Lev_\beta(\mathcal{T})$ stands for β^{th} level of \mathcal{T} .

Recall that the Boolean algebra \mathbb{B} is called (κ, λ, τ) -distributive if for every family $(\mathcal{A}_\alpha)_{\alpha < \kappa} \subseteq P(\mathbb{B}^+)$ of antichains such that $|\mathcal{A}_\alpha| \leq \lambda$ for all $\alpha < \kappa$ there is some antichain $\mathcal{B} \subseteq \mathbb{B}^+$ such that

$$|\{a \in \mathcal{A}_\alpha : a \cdot b \neq \mathbb{O}\}| < \tau \quad \text{for each } b \in \mathcal{B} \text{ and } \alpha < \kappa.$$

Define *weakly distributivity number* respectively as

$$wh(\mathbb{B}) := \min \{ \kappa : \mathbb{B} \text{ is not } (\kappa, \cdot, \omega)\text{-distributive} \}.$$

We abuse notation and write $\text{sat}(\mathcal{I}), \pi(\mathcal{I}), wh(\mathcal{I})$ etc. whenever $\mathbb{B} = P(X)/\mathcal{I}$.

In the paper we compare Boolean algebra $P(X)/\mathcal{I}$ with $C(\lambda)$ and measure algebra. The Boolean algebra $C(\lambda)$ is called *Cohen algebra* and is equivalent to forcing adding λ many Cohen reals, i.e. it includes as a dense subset the following poset (with ordering being reverse inclusion)

$$F(\lambda, \omega) = \{p : p \text{ is a function with } \text{dom}(p) \subseteq \lambda, \text{rng}(p) \subseteq 2 \text{ and } |p| < \omega\}.$$

A boolean algebra \mathbb{B} is called *measure algebra* if it carries a measure, i.e. non-decreasing real-valued non-negative function μ such that

$$\mu(\mathbb{1}) = 1, \quad \mathcal{I}_\mu := \{a \in \mathbb{B} : \mu(a) = 0\} = \{\mathbb{O}\} \text{ and } \mu\left(\sum_{n < \omega} a_n\right) = \sum_{n < \omega} \mu(a_n)$$

whenever $(a_n)_{n < \omega}$ is an antichain in \mathbb{B} . A measure μ is atomless iff \mathcal{I}_μ is an atomless ideal.

2. Results

In this section we prove several assertions on structure of the atomless Boolean algebra $P(X)/\mathcal{I}$ by the means of cardinal invariants. Most of them appeared in [3] as results of their generic ultrapower technique. Our proofs are much simpler and in the spirit of 1.12 of [1] Moreover, from combinatorial point of view, the proofs are based on the following trivial observation. Let us call a family $\mathcal{S} \subseteq \mathbb{B}^+$ an \leq -*antichain* if for all different $a, b \in \mathcal{S}$ none of $a \leq b$ or $b \leq a$ holds. Then it is easily seen that

$$\pi(\mathbb{B}) = \sup \{ |\mathcal{S}| : \mathcal{S} \subseteq \mathbb{B}^+ \text{ is an } \leq\text{-antichain} \}.$$

We begin with the slightly strengthened result from [3] concerning arbitrary atomless ideals. Before doing that observe that $P(X)/(\mathcal{I} \upharpoonright A)$ is isomorphic to $P(A)/(\mathcal{I} \cap P(A))$ for $A \in \mathcal{I}^+$. Define *hereditarily density number* of \mathcal{I} as

$$h\pi(\mathcal{I}) = \min \{ \pi(\mathcal{I} \upharpoonright A) : A \in \mathcal{I}^+ \}$$

and the following analogue for saturation called *hereditarily saturation number* given as

$$\text{hsat}(\mathcal{I}) = \min \{ \text{sat}(\mathcal{I} \restriction A) : A \in \mathcal{I}^+ \}.$$

Lemma 1

If \mathcal{I} is an atomless ideal on X then $h\pi(\mathcal{I}) \geq \text{add}(\mathcal{I})$.

Proof:

Let $A \in \mathcal{I}^+$ be arbitrary. We shall define a subtree \mathcal{T} of $2^{\leq h\pi(\mathcal{I})}$ and a mapping $\varphi : \mathcal{T} \mapsto P(A)$. Let $\emptyset \in \mathcal{T}$, $\varphi(\emptyset) = A$ and assume that levels $< \alpha$ are already constructed. If $\alpha = \beta + 1$ for some $\beta < h\pi(\mathcal{I})$ then proceed as follows. If $\varphi(t) \in \mathcal{I}^+$ for some $t \in \text{Lev}_\beta(\mathcal{T})$ then put $t^{\frown}0, t^{\frown}1$ into \mathcal{T} and define disjoint $\varphi(t^{\frown}0), \varphi(t^{\frown}1) \in \mathcal{I}^+$ subsets of $\varphi(t)$ (it is possible since \mathcal{I} is atomless); if however $\varphi(t) \in \mathcal{I}$ put only $t^{\frown}0$ in \mathcal{T} and define $\varphi(t^{\frown}0) = \varphi(t)$. For limit α put $t \in \text{Lev}_\alpha(\mathcal{T})$ if $t \restriction \beta \in \mathcal{T}$ for all $\beta < \alpha$ and for such t put $\varphi(t) = \bigcap_{\beta < \alpha} \varphi(t \restriction \beta)$. Notice that the above construction as well as the fact that \mathcal{I} is $h\pi(\mathcal{I})^+$ -saturated (no branch of \mathcal{T} can have more than $h\pi(\mathcal{I})$ splitting nodes) ensure that

$$\varphi[\text{Lev}_{h\pi(\mathcal{I})}(\mathcal{T})] \subseteq \mathcal{I} \quad \text{and} \quad A = \bigcup \varphi[\text{Lev}_{h\pi(\mathcal{I})}(\mathcal{T})].$$

Therefore

$$|\varphi[\text{Lev}_{h\pi(\mathcal{I})}(\mathcal{T})]| \geq \text{add}(\mathcal{I}).$$

Thus \mathcal{T} has at least $\text{add}(\mathcal{I})$ many branches. This implies that \mathcal{T} has at least $\text{add}(\mathcal{I})$ many splitting nodes hence we can construct inductively a disjoint subfamily of $\mathcal{I}^+ \cap \varphi[\mathcal{T}]$ of size $\text{add}(\mathcal{I})$. By the remark preceding the theorem (obviously any disjoint family is $\leq_{\mathcal{I}}$ -antichain) we conclude $\pi(\mathcal{I} \restriction A) \geq \text{add}(\mathcal{I})$. Since $A \in \mathcal{I}^+$ was arbitrary this finishes the proof. \square

Note that **Lemma 1** implies that Boolean algebra $P(X)/\mathcal{I}$ is equivalent to $C(\lambda)$ then $\lambda \geq \text{add}(\mathcal{I})$. This is slightly improved in **Corollary 3** which is immediate consequence of a theorem below.

Theorem 2 Let \mathcal{I} be an atomless ideal on X and assume that a Boolean algebra \mathbb{B} is a sum of its regular subalgebras $\{\mathbb{B}_\alpha : \alpha < \text{add}(\mathcal{I})\}$ such that $\pi(\mathbb{B}_\alpha) < \text{add}(\mathcal{I})$ for all $\alpha < \text{add}(\mathcal{I})$. Then an inequality $\text{hsat}(\mathcal{I}) \leq \text{add}(\mathcal{I})$ forbids $P(X)/\mathcal{I}$ and \mathbb{B} to be isomorphic.

Proof:

Denote $\kappa := \text{add}(\mathcal{I})$ and recall that this cardinal is regular. Without lost of generality we can assume that \mathbb{B}_α 's form increasing chain of Boolean algebras by replacing, if needed, \mathbb{B}_α by $\prod_{\beta < \alpha} \mathbb{B}_\beta \times \{\mathbb{O}, \mathbb{1}\}^{\kappa \setminus \alpha}$ and taking

coordinatewise ordering. Define $\psi : \mathbb{B} \rightarrow \kappa$ as $\psi(a) = \min\{\alpha < \kappa : a \in \mathbb{B}_\alpha\}$, $a \in \mathbb{B}$. Take arbitrary $a \in \mathbb{B}^+$. In order to prove the theorem we shall find \mathcal{I} -antichain of size κ below a . Since by **Theorem 1** we have $\pi(\mathbb{B} \restriction a) \geq \kappa$ we need only find *any* \mathcal{I} -antichain (considering $\mathbb{B}_\alpha \restriction a$ instead of \mathbb{B}_α).

We shall define a subtree \mathcal{T} of $2^{\leq \kappa}$ and an order-preserving mapping $\varphi : \mathcal{T} \mapsto \mathbb{B}$. Let $\emptyset \in \mathcal{T}$, $\varphi(\emptyset) = \mathbb{1} \in \mathbb{B}_0$ and assume that levels $< \alpha$ have already been constructed in such a way that

$$\mathcal{T} \restriction \alpha := \bigcup_{\beta < \alpha} \text{Lev}_\beta(\mathcal{T}) \subseteq \varphi^{-1}[\mathbb{B}_{\bar{\alpha}}]$$

for some $\bar{\alpha} \geq \alpha$ and $\text{Lev}_\beta(\mathcal{T})$, for all $\beta < \alpha$, is a maximal antichain in \mathbb{B} of size less than κ consisting *exactly* one splitting node. As κ is regular we get $|\mathcal{T} \restriction \alpha| < \kappa$. Since $P(X)/\mathcal{I}$ and \mathbb{B} are isomorphic then **Theorem 1** implies $\kappa \leq \pi(\mathbb{B})$. Therefore, if we choose a dense subset of $\mathbb{B}_{\bar{\alpha}}$ of size less than κ , then there exists $b \in \mathbb{B}^+$ with $\psi(b) > \bar{\alpha}$ such that $\varphi(t) - b \in \mathbb{B}^+$ for all $t \in \mathcal{T} \restriction \alpha$. We define $\text{Lev}_\alpha(\mathcal{T})$ as a subset of

$$A_\alpha := \left\{ t \in 2^\alpha : \bigcap_{\beta < \alpha} \varphi(t(\beta)) \in \mathbb{B}^+ \right\}$$

as follows. If $\mathfrak{t} \in 2^{\alpha+1}$ is the unique one such that for all $\beta < \alpha$ we have $\varphi(\mathfrak{t}(\beta)) \cdot b \in \mathbb{B}^+$ (it exists since $\text{Lev}_\beta(\mathcal{T})$'s are maximal antichains) then put \mathfrak{t}^0 and \mathfrak{t}^1 into \mathcal{T} and define

$$\varphi(\mathfrak{t}^0) = \bigcap_{\beta < \alpha} \varphi(\mathfrak{t}(\beta)) - b \text{ and } \varphi(\mathfrak{t}^1) = \bigcap_{\beta < \alpha} \varphi(\mathfrak{t}(\beta)) \cdot b.$$

For $t \in A_\alpha \setminus \{\mathfrak{t}\}$ put only t^0 in \mathcal{T} and define $\varphi(\mathfrak{t}^0) = \bigcap\{\varphi(t(\beta)) : \beta < \alpha\}$. Now, the induction hypothesis ensure that there are less than κ splitting nodes in $\mathcal{T} \restriction \alpha$; hence $\varphi[\text{Lev}_\alpha(\mathcal{T})]$ is a maximal antichain in \mathbb{B} . This completes a construction of $\mathcal{T} \restriction \kappa$: there are κ levels build as $\psi[\varphi[\mathcal{T} \restriction \kappa]]$ is unbounded in κ and κ is regular. Finally, define

$$\text{Lev}_\kappa(\mathcal{T}) := \{t^0 \in 2^\kappa : t \restriction \alpha \in \mathcal{T} \restriction \kappa \text{ for all } \alpha < \kappa\}$$

and for $t^0 \in \text{Lev}_\kappa(\mathcal{T})$ put $\varphi(t^0) = \bigcap\{\varphi(t \restriction \alpha) : \alpha < \kappa\}$. There are two cases possible. Either $|\text{Lev}_\kappa(\mathcal{T}) \cap \mathbb{B}^+| \geq \kappa$ and thus we obtain antichain of size κ , or there exists cofinal branch through \mathcal{T} with κ many splitting nodes (thanks to the construction of the tree). In the last case it is easy to find inductively antichain of size κ . This finishes the proof. \square

Corollary 3 If $P(X)/\mathcal{I}$ is equivalent to $C(\lambda)$ then $\lambda > \text{add}(\mathcal{I})$.

Similarly one can prove analogical result for measure algebras or even more general Boolean algebras. First of all note that for an atomless ideal \mathcal{I} we have $wh(\mathcal{I}) = \text{add}(\mathcal{I})$; this can be easily deduced from Ex. 14.9 of [5].

Theorem 3

If \mathcal{I} is an atomless ideal such that $wh(\mathcal{I}) \geq \text{sat}(\mathcal{I}) = \omega_1$ then $\pi(\mathcal{I}) > \text{add}(\mathcal{I})$.

Proof:

Denote $\mathbb{B} := P(X)/\mathcal{I}$ and $\kappa := wh(\mathcal{I}) = \text{add}(\mathcal{I})$. Pick a family $(\mathcal{A}_\alpha)_{\alpha < \kappa}$ of \mathcal{I} -antichains witnessing $wh(\mathcal{I}) = \kappa$. We shall construct a tree $\mathcal{T} \subseteq \omega^{<\kappa}$ and an injection $f : \mathcal{T} \rightarrow \mathcal{I}^+$ such that

- $Lev_\alpha(\mathcal{T})$ are \mathcal{I} -antichains for all $\alpha < \kappa$;
- $Lev_\beta(\mathcal{T})$ almost refines $Lev_\alpha(\mathcal{T})$ for all $\alpha < \beta < \kappa$;
- $\{t \hat{\ } n \in \mathcal{T} : f(t \hat{\ } n) \subseteq_{\mathcal{I}} f(t)\}$ is infinite for all $t \in \mathcal{T}$.

Once the construction is finished then by induction on branches of \mathcal{T} using properties of \mathcal{T} and the fact that κ is regular uncountable cardinal one build an $\leq_{\mathcal{I}}$ -antichain of size $cf(\omega^\kappa) > \kappa$. This will end the proof.

Build inductively promised \mathcal{T} and f starting with $\{\emptyset, \langle 0 \rangle, \langle 1 \rangle, \dots\} \subseteq \mathcal{T}$, $f(\emptyset) = X$ and $Lev_1(\mathcal{T}) = \mathcal{A}_1$. Let $\alpha < \kappa$ and suppose $\mathcal{T} \restriction \alpha$ and $f \restriction (\mathcal{T} \restriction \alpha)$ have been defined. Now choose an \mathcal{I} -antichain \mathcal{A}'_α almost refining simultaneously a family $\{\mathcal{A}_\beta, Lev_\beta(\mathcal{T}) : \beta < \alpha\}$; this is possible since $\alpha < wh(\mathcal{I})$. Now define $Lev_\alpha(\mathcal{T})$ in such a way that for all $A \in \mathcal{A}'_\alpha$ there exists infinite $S \subseteq Lev_\alpha(\mathcal{T})$ satisfying

$$A =_{\mathcal{I}} \bigcup_{t \in S} f(t).$$

This in turn is possible by the fact that \mathcal{I} is atomless and $\text{sat}(\mathcal{I}) = \omega_1$. One easily checks it gives mentioned construction. \square

Corollary 4 If $P(X)/\mathcal{I}$ is equivalent to forcing with λ many random reals then $\lambda > \text{add}(\mathcal{I})$.

We finish with some remarks. The use of the language of trees is not circumstantial. This is perfectly visible when comparing tree forcing notions with forcing with ideal. Boolean algebras generated by these forcings are in a sense canonical, they are the most representative and interesting among all definable forcings [7]. Still, the most intriguing problem remains: what suitable definable forcings (Boolean algebras) are not equivalent with $P(X)/\mathcal{I}$? Some results appeared in [3] for the simplest tree forcings however the problem is widely open. In forthcoming paper we use the technique from the paper to systematic study so called idealized forcings [7] and its properties preventing to be equivalent to $P(X)/\mathcal{I}$.

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RYSZARD FRANKIEWICZ

Institute of Mathematics, Polish Academy of Sciences,
Śniadeckich 8, 00-950 Warszawa, Poland
e-mail: rf@impan.pl

SŁAWOMIR SZCZEPANIAK

Institute of Mathematics, Polish Academy of Sciences,
Śniadeckich 8, 00-950 Warszawa, Poland
e-mail: szczepaniak@impan.pan.wroc.pl